

Palette Sparsification Beyond $(\Delta + 1)$ Vertex Coloring

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June 11, 2026

Problem Background: Vertex Coloring

Let $G = (V, E)$ be an undirected graph with n -vertex and maximum degree Δ .

A proper coloring assigns each vertex a color in the palette $[C] = \{1, 2, \dots, C\}$

$$\varphi : V \rightarrow [C]$$

such that for every edge $(u, v) \in E$,

$$\varphi(u) \neq \varphi(v).$$

Trivial observation: the graph is colorable by the palette when $|C| \geq \Delta + 1$.

Question: can we reduce the amount of color information stored at each vertex?

Palette Sparsification

Instead of giving every vertex the full palette $[C]$, each vertex samples a small list

$$L(v) \subseteq [C], \quad |L(v)| = \ell.$$

We ask whether, after this random sparsification, there still exists a proper list coloring:

$$\varphi(v) \in L(v), \quad \forall (u, v) \in E, \quad \varphi(u) \neq \varphi(v).$$

Palette sparsification problem

How small can ℓ be while still guaranteeing that the sampled lists are colorable whp?

Previous Work: The $(\Delta + 1)$ Case

Previous work studied the classical critical palette

$$C = \Delta + 1.$$

Each vertex samples

$$L(v) \subseteq [\Delta + 1], \quad |L(v)| = \ell.$$

Known result (Assadi, Chen, and Khanna [SODA'19]):

$$\ell = \Theta(\log n)$$

samples per vertex are necessary and sufficient for palette sparsification.

This Paper: Beyond the $(\Delta + 1)$ Case (larger and smaller)

This paper asks what happens in other coloring regimes.

Setting	Number of colors	Samples per vertex
Previous work	$\Delta + 1$	$\Theta(\log n)$
This paper	$(1 + \varepsilon)\Delta$	$\Theta_\varepsilon(\sqrt{\log n})$
This paper	$\text{deg} + 1$, locally	$\mathcal{O}(\log n)$

This presentation will cover the upper bound results of $(1 + \varepsilon)\Delta$ regime (Part 1) and $\text{deg} + 1$ regime (Part 2).

Part 1: Why Does $(1 + \epsilon)\Delta$ Need Fewer Samples?

For a fixed vertex-color pair (v, c) , define the sampled color-degree

$$\text{deg}_L(v, c) = |\{u \in N(v) : c \in L(u), \text{ i.e. } c \text{ is sampled in } u\text{'s palette}\}|,$$

which measures *how many neighbors of v may conflict with v if v uses color c .*

Original case: $\Delta + 1$

Each vertex samples from $[\Delta + 1]$. For fixed (v, c) ,

$$\mathbb{E}[\text{deg}_L(v, c)] \approx \Delta \cdot \frac{\ell}{\Delta + 1} \approx \ell.$$

So

$$|L(v)| \approx \mathbb{E}[\text{deg}_L(v, c)].$$

Relaxed case: $(1 + \epsilon)\Delta$

Each vertex samples from $[(1 + \epsilon)\Delta]$. For fixed (v, c) ,

$$\mathbb{E}[\text{deg}_L(v, c)] \leq \Delta \cdot \frac{\ell}{(1 + \epsilon)\Delta} = \frac{\ell}{1 + \epsilon}.$$

So

$$|L(v)| - \mathbb{E}[\text{deg}_L(v, c)] = \frac{\epsilon \ell}{1 + \epsilon}.$$

Part 1: Why Does Color Degree Matter?

Relaxed case: $(1 + \epsilon)\Delta$

$$|L(v)| = \ell, \text{ and } \mathbb{E}[\deg_L(v, c)] \leq \frac{\ell}{1 + \epsilon}.$$

How can color degree help coloring?

Theorem (Reed–Sudakov, informal)

For every constant $\eta > 0$, if every vertex has a list $L(v)$ such that

$$|L(v)| \geq (1 + \eta)d$$

and for every vertex-color pair,

$$\deg_L(v, c) \leq d,$$

then the graph admits a proper list coloring from $L(v)$.

Takeaway: make sampled color-degree \leq list size/ $(1 + \eta)$, then the graph is colorable.

Part 1: Why Does Color Degree Matter?

Relaxed case: $(1 + \epsilon)\Delta$

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then the graph admits a proper list coloring from $L(v)$.

Takeaway (our setting): make $\deg_L(v, c) \leq \ell/(1 + \eta)$, then the graph is colorable.

Part 1: Naive Concentration and Union Bound?

For a fixed pair (v, c) , Chernoff gives roughly

$$\mathbb{P} \left[\deg_L(v, c) \geq (1 + \Theta(\epsilon)) \frac{\ell}{1 + \epsilon} \right] \leq \exp(-\Theta_\epsilon(\ell)).$$

Consider the union bound, we have

$$\mathbb{P}[G \text{ is colorable by the sampled palette}] \geq 1 - n\ell \exp(-\Theta_\epsilon(\ell)).$$

We can only get

$$\ell = \Theta(\log n),$$

which is not enough.

Key obstacle

We cannot demand that every sampled color at every vertex has small color-degree.

- We cannot demand that every sampled color at every vertex is good.
- $\deg_L(v, c), \deg_L(v, c')$ would correlate with each other, shouldn't bound them separately.

Part 1: Change The Perspective!

Theorem (Reed–Sudakov, recall)

For every constant $\eta > 0$, if every vertex has a list $L(v)$ such that

$$|L(v)| \geq (1 + \eta)d$$

and for every vertex-color pair,

$$\deg_L(v, c) \leq d,$$

then the graph admits a proper list coloring from $L(v)$.

We can instead try to prove that, whp, for every v , only **a small fraction of colors** $c \in L(v)$ violate $\deg_L(v, c) \leq d$, then we can **delete them** from $L(v)$ while preseving $|L(v)| \geq (1 + \eta)d$.

Part 1: Defining The Bad Pairs

Definition (Bad Pairs)

Define a sampled color $c \in L(v)$ to be bad for v if

$$\deg_L(v, c) > \left(1 + \frac{\varepsilon}{2}\right) \frac{\ell}{1 + \varepsilon}.$$

Recall: for a fixed pair (v, c) , Chernoff gives roughly

$$\mathbb{P} \left[\deg_L(v, c) \geq \left(1 + \frac{\varepsilon}{2}\right) \frac{\ell}{1 + \varepsilon} \right] \leq \exp(-\Theta_\varepsilon(\ell)).$$

which is too large to use union bound.

Main trick

We do not need every vertex-color pair to be good whp. It suffices to show that each vertex has only a small number of bad colors whp.

Part 1: Bounding The Aggregation

We have

$$\mathbb{P}[c \text{ is bad for } v] \leq \exp(-\Theta_\varepsilon(\ell)).$$

Clearly, colors being bad are negatively correlated. So we have

$$\begin{aligned}\mathbb{P}[\text{at least } \varepsilon\ell/4 \text{ colors are bad for } v] &\leq \binom{\ell}{\gamma\ell} \cdot \exp(-\Theta_\varepsilon(\ell^2)) \\ &\leq \exp(\Theta_\varepsilon(\ell)) \cdot \exp(-\Theta_\varepsilon(\ell^2)) \\ &= \exp(-\Theta_\varepsilon(\ell^2)).\end{aligned}$$

So that we can set $\ell = \Theta_\varepsilon(\sqrt{\log n})$, then $\mathbb{P}[\text{at least } \varepsilon\ell/4 \text{ colors are bad for } v] \leq 1/n^{\Theta(1)}$.

Then a union bound over all vertex completes the proof. \square

Part 1: Proof Takeaway

The proof of the $(1 + \varepsilon)\Delta$ theorem:

- 1 **Color-degree:** measure conflicts per vertex-color pair, not by ordinary degree.
- 2 **Bad-color deletion:** a few high-conflict colors can be removed locally.
- 3 **Aggregation:** one bad color is not rare enough, but many bad colors at one vertex are rare enough.
- 4 **Apply Reed-Sudakov Theorem.**

Part 2: Local Palette Sparsification

In Part 1, every vertex sampled from the same global palette

$$[(1 + \varepsilon)\Delta].$$

Now we study a local model: each vertex has colors depending on its own degree.

A warm-up model

Suppose each vertex has the local palette

$$S(v) = \{1, 2, \dots, (1 + \varepsilon) \deg(v)\}.$$

Each vertex samples

$$L(v) \subseteq S(v), \quad |L(v)| = \ell.$$

Part 2: Warm-up – The $(1 + \varepsilon) \deg(v)$ Case

Simple greedy algorithm.

Process vertices in an arbitrary order. When coloring v , **at most** $\deg(v)$ **colors** have already been used by its colored neighbors.

Note that $S(v)$ has $(1 + \varepsilon) \deg(v)$ colors, so at least an $\Omega_\varepsilon(1)$ fraction of colors in $S(v)$ remain safe. Then we have

$$\mathbb{P}[\text{all safe colors for } v \text{ are not sampled}] \leq (1 - \Omega_\varepsilon(1))^\ell = \exp(-\Omega_\varepsilon(\ell)).$$

Taking

$$\ell = O_\varepsilon(\log n)$$

makes this probability small enough to union bound over all vertices.

Part 2: The Harder Local Case $\deg(v) + 1$

Local sparse palette model

Now consider a harder local palette

$$S(v) = \{1, 2, \dots, \deg(v) + 1\}.$$

Each vertex samples

$$L(v) \subseteq S(v), \quad |L(v)| = \ell.$$

Clearly the simple greedy algorithm doesn't work now, since we cannot guarantee the existence of safe colors.

Can we prove that whp G is colorable by the local sparse palette with $\ell = \mathcal{O}_\varepsilon(\log n)$?

Part 2: Proof Strategy

Main proof idea

We can create extra available colors, so that we can then apply the greedy algorithm.

Where does the extra available color come from?

Two cases:

Type	Meaning	Source of hidden slack
Sparse vertices	Many $u \in N(v)$ are non-adjacent	Non-adjacent neighbors can share a color
Uneven vertices	Many $u \in N(v)$ are with much higher degree	High-degree neighbors may use colors outside $S(v)$

We will first handle vertices of these two types.

Part 2: Proof Strategy

Randomized partial coloring

- 1 Activate each vertex w.p. $\Theta(\varepsilon^2)$.
- 2 Each active vertex sequentially tries one random color from its sampled list, and keep it if feasible.

Step 1: After this randomized partial coloring, show that every remaining sparse/uneven vertex satisfies

$$\underbrace{|S_1(v)|}_{\text{colors still available}} \geq \underbrace{\deg_1(v)}_{\text{uncolored neighbours}} + \underbrace{\eta \deg(v)}_{\text{safe colors}}.$$

Step 2: Greedy coloring. When coloring v , at most $\deg_1(v)$ colors are blocked by already colored neighbors. Therefore at least $\eta \deg(v)$ safe colors. Similarly we have

$$\mathbb{P}[\text{all safe colors for } v \text{ are not sampled}] \leq (1 - \eta)^{\ell-1}.$$

Then directly apply $\ell = O(\log n)$ and a union bound completes the coloring of these vertices.

Part 2: Step 1 for Uneven Vertices

Let v be an uneven vertex, *i.e.*, v has many neighbors u with $\deg(u) \gg \deg(v)$.

(ϵ -uneven \iff for $\geq \epsilon \deg(v)$ neighbors u of v , $\deg(v) < (1 - \epsilon) \deg(u)$)

The neighbor u has many colors outside v 's palette:

$$S(u) \setminus S(v) = \{\deg(v) + 2, \dots, \deg(u) + 1\}.$$

Concentration claim

For an uneven vertex v , whp there are $\eta \deg(v)$ high-degree neighbors successfully colored outside $S(v)$ after the randomized partial coloring.

Each such successful neighbor contributes one unit of slack:

$$\underbrace{|S_1(v)|}_{\text{colors still available}} - \underbrace{\deg_1(v)}_{\text{uncolored neighbours}} \quad \text{increase by 1.}$$

Hence whp

$$|S_1(v)| \geq \deg_1(v) + \eta \deg(v)$$

for every uneven vertex v after a union bound.

Part 2: Step 1 for Sparse Vertices

Let v be a sparse vertex, *i.e.*, $N(v)$ contains many non-edges $u_1, u_2 \in N(v), (u_1, u_2) \notin E$.
(ϵ -sparse \iff contains $\geq \epsilon^2 \binom{\deg(v)}{2}$ non-edges)

Concentration claim

For a sparse vertex v , whp there are $\eta \deg(v)$ colors retained by two non-adjacent neighbors of v after the randomized partial coloring.

Each such successful sharing event contributes one unit of slack:

$$\underbrace{|S_1(v)|}_{\text{colors still available}} - \underbrace{\deg_1(v)}_{\text{uncolored neighbours}} \text{ increase by 1.}$$

Hence whp

$$|S_1(v)| \geq \deg_1(v) + \eta \deg(v)$$

for every sparse vertex v after a union bound.

Part 2: Step 1 for Sparse Vertices (cont'd)

Let v be a sparse vertex, *i.e.*, $N(v)$ contains many non-edges $u_1, u_2 \in N(v), (u_1, u_2) \notin E$.
(ε -sparse \iff contains $\geq \varepsilon^2 \binom{\deg(v)}{2}$ non-edges)

Let

$$Z = \#\{\text{colors retained by two non-adjacent neighbors of } v\}.$$

The goal is to prove $Z \geq \eta \deg(v)$ whp. We can split Z as:

$$Z = T - D,$$

where T counts colors sampled by two non-adjacent neighbors, and D counts those killed by conflicts.

Concentration tool

T and D are Lipschitz and certifiable, so Talagrand's inequality gives concentration. Therefore $Z = T - D \geq \eta \deg(v)$ whp.

Hence

$$|S_1(v)| \geq \deg_1(v) + \eta \deg(v).$$

Part 2: Coloring Remaining Vertices

Key idea: vertex set decomposition

If a vertex is neither sparse nor uneven, then it is in an almost-clique.

$$V = V_{\text{sparse}} \sqcup V_{\text{uneven}} \sqcup K_1 \sqcup \dots \sqcup K_m.$$

- V_{sparse} : vertices whose neighborhoods have many missing edges;
- V_{uneven} : vertices with many much higher-degree neighbors;
- K_i : almost-cliques, where the local structure is densely connected and the degrees are even.

How to handle almost-cliques?

High-level idea

Inside an almost-clique K , reduce the local palette $S(v) = \{1, \dots, \deg(v) + 1\}$ to a global palette $\{1, \dots, \Delta(K) + 1\}$. Colors larger than $\deg(v) + 1$ are ignored.

So we can directly use the previous result on global palette $\mathcal{O}(\log |K|)$.

Part 2: Proof Takeaway

- 1 **Decompose the graph** into sparse vertices, uneven vertices, and almost-cliques.
- 2 **Use random partial coloring** to create extra colors for sparse and uneven vertices.
- 3 **Reduce to greedy coloring** once

$$|S_1(v)| \geq \deg_1(v) + \eta \deg(v).$$

- 4 **Handle almost-cliques** using previous global palette result.

Summary

- **Problem.** Palette sparsification asks how many random colors each vertex needs to keep.
- **Part 1.** The key idea is to delete a small number of bad colors rather than requiring every color to be good.
- **Part 2.** The key idea is to decompose the vertex set (sparse, uneven, almost-clique), and use concentration to squeeze extra colors from sparse and uneven vertices.